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Some fundamental errors in regard to the nature and origin of number have been given such wide circulation in recent works on psychology and the psychology of number that it seems desirable to present views which are based upon the scientific and revolutionary work upon the nature of the number idea which has been done by Cantor,* Dedekind, Peano, Frege, and others. Psychologists, philosophers, and pedagogs have treated the subject of the definition of number and the genesis of the number idea in apparent ignorance of the fact that great mathematicians have also labored in this field. While some may dispute the right of mathematicians to discuss so psychological a matter as the genesis of the number idea, no one can dispute that a somewhat logical definition of number is a necessary basis for work on the genesis of the number idea.

One faulty definition which persists is that number is ratio. If that be so, pray what is ratio? Surely the idea of ratio is no more evident than is that of number.

Upon this poor foundation a method of teaching arithmetic is based whose merit, if any, is in spite of this fundamentally worthless concept of number. To confuse number with ratio is to confuse number with an application of number. Logically ratio is an unsymmetric relation between two numbers.

Some writers would have us believe that number takes its origin in counting. Here, again, we may ask, is counting intuitive? Is the idea of counting any simpler than the idea of number? Counting presupposes the presence of the number idea in the mind of the counter. Children often go through the counting process in much the same way that they repeat Old Mother Goose jingles; as much number concept is present in one operation as in the other. The same criticism holds against that view of number

* Cantor, Georg, *Mathematische Annalen*, Vol. XLVI, p. 481, *Beitrage zur Begründung transfiniten Mengenlehre.* Dedekind, R. *Was sind und was sollen die Zahlen?* Also translated into English by W. W. Beman in *Essays on the Theory of Numbers*, Open Court Co. Frege, G. *Grundlagen der Arithmetic.*

which regards the number idea as evolved in some mysterious way by successive strokes of attention.

The need for measuring is given by a very eminent philosopher as the determining factor which evolves number. Necessity may be the mother of invention in a general way, but certainly not of the invention of the idea of number. The slightest reflection shows that measuring involves having number concepts as a basis.

It would seem out of place to refer to those writers who regard numbers as objects. However, there yet exists schools in America and Germany in which the children ostensibly learn all about the number four before proceeding to the number five.

In all of these we have the applications of number confounded with the actual number idea. Most eminent mathematicians and philosophers have made such errors. Euler defined number as ratio, and the substance of the definition by Leibnitz is that number is obtained by comparing a given magnitude with a unit magnitude of the same kind. However, neither did these men nor did any of the great mathematicians of more than one hundred years ago occupy themselves at all deeply with the definition of number. They assumed number as being *a priori* and a knowledge of the fundamental operations with integers and fractions as presupposable. Within the last fifty years the philosophical bases of the science of geometry and analysis have been carefully studied. In geometry out of these studies came the non-Euclidean geometries. In analysis the result has been to show that logic and mathematics are fundamentally sister studies, in fact almost Siamese twins.

A logical definition of cardinal number presupposes the idea of a group or class of objects and the notion of membership of a group. It also involves the idea of one to one correspondence; that is, the definition implies that we know what is meant when it is stated that each child has an orange, "to each child an orange." Groups of objects are in one to one correspondence when to each object of one group there corresponds one and only one object of the other group, and vice versa. Upon the basis of these ideas we may say that that concept which is common to all groupes of objects which may be placed in one to one correspondence with the objects of any given group is called the number of the given group. This definition is essentially the same as stating that the number of a group of objects is that concept which remains when we abstract from the particular objects which make up the group any special characteristics of those objects as well as any notion of order.*

It is not to be denied that logical objections may be made to these definitions, but the advance is so great over the older ideas that any modern attempt at a psychology of number must take account of this work.

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* These two definitions are in substance given by G. Cantor in the article noted above. Dedekind and Frege also use the one-to-one correspondence idea.

Preceding the acquisition of definite number concepts is the acquisition of the ideas involved by the words, "another," and "more," used quantitatively and also as equivalent of "others," and the perception of groups of like objects as groups. The group idea is constantly brought to the child's attention by the use of the plural—apples, pears, balls, etc. This forms the important step in the evolution of the number idea, of perceiving likeness in different objects and of naming that quality.

At some stage in the child's development, according to Perez and others usually about the age of two years, the child is ready to acquire a name for that special group characteristic which is constantly presented to him in his own hands, feet, eyes, and ears. The child delights in the word two as emphasizing a new idea and seeks application for it. Two apples, two oranges, two pears, two ladies, are combinations that readily follow upon the presentation of the objects. Even yet there may be confusion of groups of two with groups of greater number, but the great step is made of a separate name to denote that property of combination of like objects into one group.

The child experiences a sense of great pleasure in the acquisition of this new idea. The feeling of pleasure is analagous to that experienced by an adult on attaining a new mental experience such as going up in a balloon or descending into a large underground cave. Pierre Loti refers to a similar feeling of exhiliration which he experienced as a young child on discovering that he could jump.

At about this point the child's number system consists of object, two, many. The base of this number system is the number two. Primitive people exist who have not attained any farther than this number system. The binary system of numbering is almost universal among the tribes of Aborigines in Australia* and is common among the tribes of South America. These tribes have words for one, two. Three is given as two and one; four as two and two; five as two, two, one, or sometimes one, two, two, and six as two, two, two. Usually the system does not go much farther than this.

Following two and many as more than two, or possibly almost accompanying these ideas, is the idea of unity. While logically the definition of unity rests on identity, psychologically the concept of unity is that of membership of a group. The child does not easily speak of "one" mother, whereas "one apple" is quite easily achieved.

Three follows very readily as two and one,† and four as two-two. There is no logical nor psychological reason why "three" need appear before "four," nor "two" before "three" in the child mind. Indeed, if the groups of objects most constantly brought before the child's attention by his own

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* Mathew, John, *Eaglehawk and Crow, a Study of Australian Aborigines.*

† In this connection it is striking that the root of three, "tri," signifies "more than," meaning one more than the two preceding numbers. Given by Bopp, *Grammaire comparée.*

body were groups of threes, that would be the first fundamental group, and three would be the first number appreciated.

Experimental psychologists find that groups of one, two, three or four objects are immediately perceived as to number, that is, that the time required to recognize that there are four objects in a group is not apparently longer than that there are two or three, whereas to group five or six objects takes a longer time unless these objects are favorably arranged. In this psychological fact we may have the reason why many primitive people do not get beyond the number four. We can only conjecture that there is some connection between this and the fact that we unconsciously have a four group, our fingers, constantly thrust before us.

Five comes as a second natural number base. It is either the base or subsidiary base of practically all number systems beyond the binary stage. The Roman Numerals are striking illustrations of the use of five as a secondary base, as the system is a ten system. The same is true of the Attic system or numerals in use for many centuries among the Greeks. In this system

1=I	5=Γ
10=Δ	50=$\Gamma\Delta$
100=H	500=Γ_{H}
1000=X	5000=Γ_{X}
10000=M	50000=Γ_{M}

Among the Mayas who have a twenty system the five also comes in as a subsidiary base. Possibly the rhythm may account for the appearance of five and multiples of five in so many of the children's counting games. In *Pädagogische Studien* Dr. E. Wilk has emphasized the fundamental importance of the five system in a series of articles on a new number method based on the natural origin of number and reckoning. Dr. Wilk states: "It is the most important result of my investigation concerning the origin of numbers that these can be formed only by the introduction of a number system." To this end he names the use of the fingers as the best material for the early work. In our schools we have undoubtedly gone too far in banishing the fingers from the early number work, but best material for number work does not exist just as best food for children does not exist. Especially with weak-minded children and children slow to grasp number facts the finger reckoning is of vital importance. Attention has recently been called to this in the *Zietschrift für Kinder forschung* (with special reference to pedagogical pathology) by Dr. H. Noll in an article on finger activity and finger reckoning as aids in developing the intelligence and the reckoning ability in weak-minded children.

Six, seven, eight and nine come as five and one, five and two, five and three, five and four. No difficulty attaches to developing ten as a new unit

as the child's ten fingers stare him in the face and force ten as a unit. Our system of money also greatly facilitates the acquisition of ten as a unit, as most children have all too intimate acquaintance with pennies and nickles and dimes. Probably, too, the money forms as convenient material as any, and more impressive than most, for developing the ideas of eleven, twelve, up to nineteen, and for twenty, thirty, forty, etc. The universal ten system among civilized peoples is due, as Aristotle first pointed out, to the fact that we have ten fingers. Certain mathematicians and even psychologists have argued that a twelve system would be better adapted to human needs, and some feeble attempts have been made to institute such a system. Logically and mathematically twelve would be a better system for adults, as the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ would be .6, .4 and .3 respectively, meaning $\frac{6}{12}$, $\frac{4}{12}$ and $\frac{3}{12}$. Unfortunately we are not logical beings but psychological, bound hand and foot (by fingers and toes) to the decimal system. Whether this influence of the fingers is subconscious or conscious, cannot be definitely shown, but the number systems of all civilised races show that this influence is the most powerful factor in the forming of a number system.

To summarize, the steps in the development of definite number ideas would seem to be as follows:

(1). Many, another, more. Perception of likeness in other individuals, also noting of plurals.

(2). Two.

(3). Many, as more than two.

(4). Two and one. Two as a number base.

(5). Two and two.

(6). Five. as two, two and one, or four and one. Five as a number base.

(7). Five and one, five and two, five and three, five and four.

(8). Ten as two fives, a new unit.

(9). 11-20 as ten and one up to two tens.

(10). 20, 30, 40, 50, 60, etc., as 2, 3, 4, 5, ... 10 tenf.

The application of these ideas to the teaching of arithmetic is immediate. In the earliest number work it means continued emphasis on fundamental group ideas. Two will always be associated with the hands, four with the fingers without the thumb, five with the fingers of one hand, seven with the days of the week, ten with the fingers of two hands, and other numbers with characteristic groups in the school room, *e. g.*, if there happen to be six panes of glass in the window, six will be associated with the number of window panes. The earliest number concepts do not come from the consideration of mathematical objects such as cubes and squares nor even splints. By presenting these first the development of the number idea is retarded as the child mind is required to struggle with these unfamiliar objects. To the child these geometrical objects have no meaning; he does not see why he should observe them. Much easier is it for him to get the

number phase from objects with which he is familiar. In fingers and apples he has an interest and any new ideas about these old friends are seized with some avidity.

To question the child's power to abstract the difference of like objects is to question his power of imagination. that prime requisite of a mathematician which power the child has most preeminently.

For the further work in arithmetic this development means the constant and recurring emphasis of the decimal system. Eighteen hundred years ago Nikomachus of Gerasa, the father of Arithmetic, did better than we in giving a table of multiples only up to 9×9, and the earliest printed arithmetics used the ordinary multiplication table only to 9×9 or 10×10. By so doing is emphasized the use of the decimal system, as all further multiplicative combinations involve only these fundamental number of facts. With the addition tables, too, there needs to be emphasis on the system. Here we may see one great way to simplify our arithmetic, namely, to play the system.

A SOLUTION OF THE BIQUADRATIC EQUATION.

By LEROY A. HOWLAND, Middletown, Connecticut.

The following solution of the biquadratic rests upon the analytic operations suggested by geometrical considerations. No assumption is made as to the form of the solution. Each step in the solution and in the discussion of multiple roots has its geometrical analogon. The cubic resolvent has a geometrical meaning and its discriminant is at the same time the discriminant of the biquadratic. M. Fritz Hofmann has used the degenerate members of a family of conics through four points to determine the roots of the biquadratic,* but in an entirely different way. He does not appear to have extended his method to a discussion of the nature of the roots.

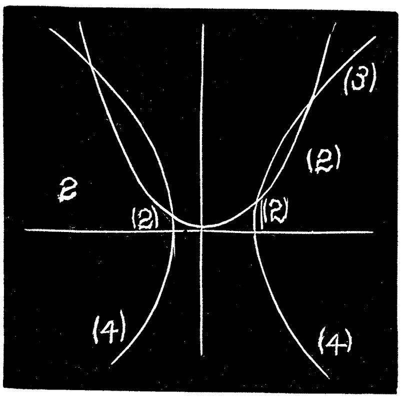

Fig. 1.

1. Solution. Let the biquadratic be

(1) $$a_0z^4+4a_1z^3+6a_2z^2+4a_3z+a_4=0 \quad (a\neq0)$$

This goes over by the substitution $z=\dfrac{x-a_1}{a_0}$ into

(2) $$x^4+ax^2+bx+c=0,$$

* *Nouvelles Annales de Mathématiques, Troisième Série, 7.*